Counting in two ways

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Week 7

1 Theory

Being able to employ a combinatorial point of view in seemingly non-related problems may often prove very useful - even provide a proof. In the following, we will take a look at a number of identities that can be proved by counting in two different ways. The simple example of usefulness of this technique can be the Handshaking lemma.

Example 1 (Handshaking lemma)**.** *For any undirected graph, we have* $\sum_{v \in V} deg(v) = 2|E|$.
(where we are summing over all vertices v*,* $deg(v)$ *is the number of edges connected to vertex* v *and* $|E|$ *is the number of edges in the graph)*

Proof. We will prove this lemma by counting in two ways. The number of edge-vertex connections is $\sum_{v \in V} deg(v)$ when we take a look at the connections of each vertex. At the same time, if we take a look at the connections of each edge, the number of edge-vertex connections is $2|E|$. □

Let us now define the following, which are most probably very familiar to you, combinatorially.

Definition 1 (Binomial numbers)**.** $\binom{n}{k}$ *is the number of ways how to choose* k *elements out of* n*.*

Definition 2 (Fibonacci numbers)**.** F_n *is the number of ways to fill a table of size* $(n-1) \times 1$ *by tiles of size* 1×1 *and* 2×1*.*

Exercise 1. *Convince yourself that the above definitions agree with the usual definitions of a binomial and Fibonacci numbers.*

2 Problems

Problem 1. *See that the number of subsets of a set with* n *elements is* 2^n*.*

Problem 2. *See that* $\binom{n}{k} = \binom{n}{n-k}$*.*

Problem 3. *See that* $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$*.*

Problem 4. *See that* $n\binom{n-1}{k-1} = k\binom{n}{k}$*.*

Problem 5. *See that* $\sum_{i=1}^{n} i = \binom{n+1}{2}$*.*

Problem 6. *See that* $\binom{n}{r} = \binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r}$*.*

Problem 7 (Binomial theorem)**.** *See that* $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$*.*

Problem 8. *See that* $\sum_{i=1}^{n} i^2 = 2\binom{n+1}{3} + \binom{n+1}{2}$*. Can you derive a formula for* $\sum_{i=1}^{n} i^3$*?*

Problem 9. *See that* $\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 2^{2n}$*.*

Problem 10. *See that* $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$*.*

Problem 11. *See that* $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

Problem 12. *See that* $F_n + F_{n+1} = F_{n+2}$.

Problem 13. *See that the number of ways to fill a table* $(n-1) \times 2$ *by tiles of size* 2×1 *is* F_n.

Problem 14. *See that* $F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b$.

Problem 15. *See that* $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$.

Problem 16. *See that* $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} - 1$.

Problem 17. *See that* $nF_0 + (n-1)F_1 + \cdots + F_{n-1} = F_{n+3} - (n+3)$.

Problem 18. *See that* $F_{2n+1} = F_n^2 + F_{n+1}^2$.

Problem 19 (Number of divisors)**.** *See that for number* n *with prime factorization* $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ *the number of its divisors is* $(a_1+1)(a_2+1)\cdots(a_k+1)$.