Barycentric Coordinates

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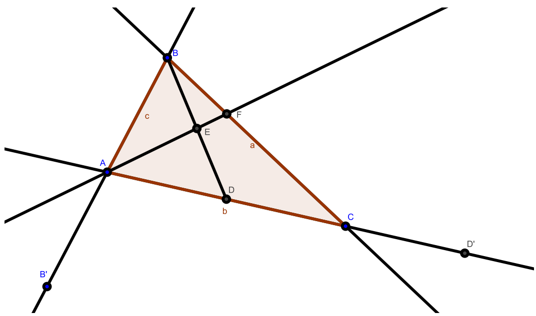
Last week we have used weights to solve some tiling problems by especially using the centroid of different points. Interestingly there is a different very beautiful application of weights in Geometry which is known as the method of *Barycentric coordinates*. We are all used to working with cartesian coordinates which involve assigning coordinates (mostly 2 or 3) to a point. We all know that this gives abstract geometric problems a very basic setting and a good way to tackle them computationally. The problem that often appears are the difficult coordinates of certain points which make calculations very tedious.
Barycentric coordinates really try to overcome this problem by not starting with axis but with vertices as the foundation for the coordinates. Let us start with the example of three points which form a triangle Every point P is given by $\vec{P} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C}$ with $\alpha + \beta + \gamma = 1$ where α, β, γ are called weights. As an example we have:



$$\vec{A} = 1 * \vec{A} + 0 * \vec{B} + 0 * \vec{C}, \vec{D} = \frac{1}{2} * \vec{A} + \frac{1}{2} * \vec{B} + 0 * \vec{C}$$

It is sometimes helpful to visualise those weight by thinking about a needle at the point P and we want to balance the plane on this needle. The question is which weights we need to place at A, B and C so that the plane is in a stable state. Let's start with a few problems to get used to barycentric coordinates.

1. Assume you are given the barycentric coordinates of P and Q. What are the coordinates of the midpoint of PQ?
2. Find the coordinates of the remaining points in Figure .
3. Find the Barycentric coordinates of the centroid of a triangle.
4. Show that the medians of a triangle divide each other in the ratio $1:2$.
5. What is the ratio of the areas of triangles that appear?
6. Barycentric coordinates are sometimes called *aereal coordinates*. Do you see why?

It seems very intuitive that barycentric coordinates are really useful in tasks which involve ratios of areas or lengths. Two very powerful theorems involving these are Ceva's and Menalaus' theorem. They are hard to prove using Euclidean geometry but appear much simpler with Barycentric coordinates. However the proofs are still not trivial, hence I added also a few other problems:

7. Find the coordinates of the incenter of a triangle.
8. How do these coordinates relate to the way we usually find the incenter of a triangle?
9. Find the equation of a line a Barycentric coordinates (hint: Start with a line through one of the vertices).
10. Prove Ceva's theorem: Let O be any point within i triangle ABC. Then draw the lines through O and the vertices of the triangle and name the intersections with the opposite edges D, E and F respectively, then
$$\frac{AF}{FB} * \frac{BD}{DC} * \frac{CE}{EA} = 1$$
11. Prove Menalaus' theorem: Let l be a line which is intersecting the sides of a triangle ABC (possibly outside the edges) then:
$$\frac{AF}{FB} * \frac{BD}{DC} * \frac{CE}{EA} = (-)1.$$
There is a minus sign because this theorem also holds for directed length but this is not hugely important today.

There are other very remarkable results using the barycentric coordinates also in higher dimension but for today we just leave it with a few more problems. Note that it sometimes useful to introduce more than 3 weights but it obviously does not change any of the theory we discussed.

12. We are given three glasses of water with capcities 3, 5 and 8 oz. How can we measure 4 oz with those. Try to solve this using barycentric coordinates.
13. Three jugs are given, each containing an integer amount of pints of water. It is allowed to pour in any jug as much water as it already contains, from any other jug with a greater amount of water. Prove that after several such pourings it is possible to empty one of the jugs. (Assume that the jugs are sufficiently large; each capable of holding all the water available.)
14. Let there be a 3D quadrilateral $A_1A_2A_3A_4$ with the property that all of its 4 sides, A_1A_2, A_2A_3, A_3A_4, A_4A_1, are tangent to a given sphere. (The side A_{ij} touches the sphere at point T_{ij}.) Interestingly, the four points of tangency $T_{12}, T_{23}, T_{34}, T_{41}$ are necessarily coplanar.
15. (USA TST 2003/2) Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, PC intersect sides BC, CA, AB at D, E, F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)
16. (ISL 2005/G1) Given a triangle ABC satisfying $AC + BC = 3 * AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.
17. (ISL 2001/G6) Let ABC be a triangle and P an exterior point in the plane of the triangle. Suppose the lines AP, BP, CP meet the sides BC, CA, AB (or extensions thereof) in D, E, F, respectively. Suppose further that the areas of triangles PBD, PCE, PAF are all equal. Prove that each of these areas is equal to the area of triangle ABC itself.