The Cauchy-Schwarz Inequality

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October 1, 2017

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Named after mathematicians Augustin-Louis Cauchy and Hermann Amandus Schwarz, the Cauchy-Schwarz inequality is a fundamental result in the theory of inequalities and has a wide range of applications and extensions in as diverse fields as number theory, analysis, probability theory, vector algebra, and many others.

You might be familiar with the quite different looking $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$ from high school or Linear Algebra, but the two inequalities are actually very closely related!

Problem 1: If $\vec{u} = (a_1, a_2, ..., a_n)$ and $\vec{v} = (b_1, b_2, ..., b_n)$, find the relationship between the two inequalities.

Problem 2: See if you can prove the triangle inequality, $|\vec{u}+\vec{v}| \leq |\vec{u}|+|\vec{v}|$, using the Cauchy-Schwarz inequality. (Note that as opposed to many geometric proofs of this inequality, this proof holds for any number of dimensions!).

Problem 3: The C.-S. inequality is also useful to justify the definition of an angle between two vectors ($\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$). Can you see why?

General proof:
Concisely,

$$0 \leq (a_1x+b_1)^2+(a_2x+b_2)^2+...+(a_nx+b_n)^2 = \left(\sum a_i^2\right)x^2+2\left(\sum a_ib_i\right)x+\left(\sum b_i^2\right).$$

Noticing that this polynomial has at most one real root (why?), we get that the discriminant $4\left(\sum a_i b_i\right)^2 - 4\left(\sum a_i\right)^2\left(\sum b_i\right)^2 \leq 0$, hence

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

This neat proof is due to Augustin-Louis Cauchy (1821).

Now, a few warm-up exercises to get used to the inequality:

Problem 4: Show that $\sum_{i=1}^{n} a_i \leq \sqrt{n}\left(\sum_{i=1}^{n} a_k^2\right)^{1/2}$.

Problem 5: Show that if $\sum_{i=1}^{n} a_i = 1$, then $\sum_{i=1}^{n} \frac{1}{a_i} \geq n^2$. When is the lower bound reached?

Problem 6: Let x_i be a sequence of unique positive numbers, and let y_i be some reordering of x_i (For those familiar with permutations, this means that $y_i = x_{\sigma(i)}$). Find an upper bound for $\sum_{i=1}^{n} x_i y_i$.

The next problems might be more difficult, but give useful and interesting extensions and applications of the C.-S. inequality.

Problem 7: Show that $\left(\sum_{i=1}^{n} a_i b_i c_i\right)^4 \leq \left(\sum_{i=1}^{n} a_i^2\right)^2 \sum_{i=1}^{n} b_i^4 \sum_{i=1}^{n} c_i^4$.

Problem 8: Show that $\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} c_i^2$.

Problem 9: If you are familiar with Euler's most famous results, show that $\sum_{i=1}^{n} \frac{a_i}{i} < \sqrt{2}\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$.

Problem 10: Show that $\sum_{i=1}^{n} \binom{n}{i} a_i \leq \binom{2n}{n}^{\frac{1}{2}} \left(\sum_{i_1}^{n} a_i^2\right)^{\frac{1}{2}}$.

Challenge: Find your own proof of the Cauchy-Schwarz inequality!

A Renaissance coda: First proven by Pietro Mengoli (1625-1686) using simple algebraic techniques (hence not our C.-S. ineq.), the following inequality was used to find one of the first proofs that $H_\infty = \lim n \to \infty \sum_{i=1}^{n} \frac{1}{n}$ diverges:

$$\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x}$$

if $x > 1$. Try to prove the inequality, and also that H_∞ converges.