EXPLORING FRACTIONS
Continued fractions

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Introduction

We all know fractions from a very early age. We also know that not any real number can be written as a fraction (although you probably got to know this a tad later) - there is the set of irrational numbers for which it is not possible. Today, we will introduce an interesting concept of continued fraction, which somehow extends the concept to all of the real numbers. Not only will every real number have a representation as a continued fraction, these will also provide us with a straightforward way to *approximate* the irrationals.

Anyway, what do we actually mean by fraction? Although we will not dig deeper into this, for completeness, let me include the modern formal definition of fractions:

Definition 1 (Fraction)**.** A fraction is an ordered pair of integers (a, b), $a \in \mathbb{Z}$, $b \in \mathbb{N}$. We say that $(a, b) = (c, d)$ if $ad = bc$, and the usual operations are defined by the following rules:

$$(a, b) \pm (c, d) = (ad \pm bc, bd)$$
$$(a, b) \cdot (c, d) = (ac, bd)$$
$$(a, b) : (c, d) = (ad, bc)$$

Continued fractions

Before we will start with continued fractions, we will need to introduce *commensurability.*

Definition 2. Two non-zero real numbers a, b are said to be **commensurable** if $\frac{a}{b}$ is a rational number.

Exercise 1. Are $\sqrt{2}$ and $\sqrt{200}$ commensurable? What about $\sqrt{2}$ and $\sqrt{5}$? What about $\sqrt{8}$ and $\sqrt{2} + 1$?

You might have heard about **Euclidean algorithm** to find the greatest common divisor of two integers, but you may not know that is it possible to apply it to any two commensurable numbers. Let me include here my favourite form of the Euclidean algorithm. Given two numbers, p and q, follow the algorithm:

(1) If $\min(p, q) = 0$, terminate and return $\max(p, q)$. This will be the "greatest common divisor"[1] of p, q.

(2) Reassign $p = q$, $q = p \bmod q$ and go to step (1).

Exercise 2. Will the above algorithm terminate if p and q aren't commensurable? Justify your conclusion.

Now we know enough to gain a good intuition about continued fractions. Consider the quadratic equation $x^2 - 3x - 1 = 0$ and let's attempt to solve it. We may divide the whole equation by x, which would give us:

$$x = 3 + \frac{1}{x}$$

[1] or at least, it's analogue in commensurable numbers, rather than integers

In order to solve for x, we would like to get rid of the x in the denominator. But we know already that $x = 3 + \frac{1}{x}$! Hence we may continue in the following manner:

$$x = 3 + \frac{1}{x} = 3 + \cfrac{1}{3 + \cfrac{1}{x}} = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{x}}} = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{x}}}} = \cdots$$

This gives us the sequence:

$$3,\ 3 + \frac{1}{3},\ 3 + \cfrac{1}{3 + \cfrac{1}{3}},\ 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3}}},\ \cdots$$

Exercise 3. Does the above sequence converge? If so, what does it converge to? (You only need to answer this, we will attempt to prove this later)

Exercise 4. What happens with the second root of the equation?

Definition 3 (Continued fraction)**.** A **continued fraction** is an (possibly infinite) expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}}$$

If we have $b_i = 1$ for all i, we will say that the continued fraction is **simple**. We will also use a special notation for the simple continued fractions: $[a_0; a_1, a_2, a_3, \dots]$.

Exercise 5. How does the root of the quadratic equation that we have just encountered look like in this notation?

Exercise 6. Think of your favourite rational number with denominator greater than 100 and nominator greater than nominator. What does your number look like in this notation?

Exercise 7. Apply the Euclidean algorithm on the nominator and denominator of your favourite rational number from the previous exercise. Do you see any connections?

Exercise 8. Are continued fractions unique? Are simple continued fractions unique?

Problem 1. Show that the continued fraction representation of a real number is finite if and only if it is rational.

Problem 2. Find the simple continued fraction representation of ϕ (the Golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$).

Problem 3. Think of a few of your favourite irrational square roots of a number. What do their continued fraction representations look like? Prove that any number x has a periodic continued fraction expansion if and only if it is an irrational square root.

Problem 4. Show that

$$[a_0; a_1, a_2, \dots, a_n, 1] = [a_0; a_1, a_2, \dots, a_{n-1}, a_n + 1]$$

Problem 5. Show that

$$\frac{1}{[0; a_0, a_1, a_2, \dots]} = [a_0; a_1, a_2, \dots]$$

Definition 4 (Convergents)**.** For a given continued fraction $[a_0; a_1, a_2, \dots]$ (from now on, we will assume that this continued fraction represents a fixed irrational), we will call the terms of the sequence $[a_0;], [a_0; a_1], [a_0; a_1, a_2], \dots$ **convergents** of the continued fraction.

Let's now take a look how the convergents are related to each other. Intuitively, given a few consequent convergents, we should be able to determine the next one - this indeed turns out to be the case, which allows us to find the approximations more easily.

Consider a given continued fraction $[a_0; a_1, a_2, \dots]$ and let n_k, d_k denote the numerator and denominator of the k-th convergent (after it is simplified and has both integral nominator and integral denominator). Then we have:

$$n_k = a_k n_{k-1} + n_{k-2} \quad \text{and} \quad d_k = a_k d_{k-1} + d_{k-2}$$

where $n_{-1} = 1$, $n_{-2} = 0$, $d_{-1} = 0$, $d_{-2} = 1$.

Problem 6. Show that this is true.

This observation gives raise to some very useful properties of the convergents. Namely:

Problem 7. Show that $n_k d_{k-1} - n_{k-1} d_k = (-1)^k$.

Problem 8. Deduce that each of the convergents $\frac{n_k}{d_k}$ is in its simplest form.

Problem 9. Show that

$$\frac{n_k}{d_k} - \frac{n_{k-1}}{d_{k-1}} = \frac{(-1)^{k+1}}{d_k d_{k-1}}$$

Problem 10 (Continued fraction as an alternating infinite series)**.** Persuade yourself that

$$[a_0; a_1, a_2, \dots] = a_0 + \sum_{k=0}^{\infty} \frac{(-1)^k}{d_k d_{k+1}}$$

We are in the position now to notice a remarkable property of the convergents. Observe on any continued fractions expansion of an irrational number that the following holds:

$$\frac{n_0}{d_0} < \frac{n_2}{d_2} < \frac{n_4}{d_4} < \dots < \frac{n_5}{d_5} < \frac{n_3}{d_3} < \frac{n_1}{d_1}$$

But this means that an irrational number is being squeezed between the even-indexed convergents and odd-indexed convergents!

Problem 11. Show that for all k we have $\frac{1}{d_k d_{k+1}} > \frac{1}{d_{k+1} d_{k+2}}$.

Remark 1. The sum of the first k terms in $\sum_{k=0}^{\infty} \frac{(-1)^k}{d_k d_{k+1}}$ is just $\frac{n_k}{n_d}$.

Whether you have encountered alternating series in Analysis or you are only about to, the theory[2] tells us that the proof of the preceding problem is enough to tell that $A = [a_0; a_1, a_2, \dots] = a_0 + \sum_{k=0}^{\infty} \frac{(-1)^k}{d_k d_{k+1}}$ converges.
And not only that, it also gives[3] the following useful inequality:

$$\left| A - \frac{n_k}{d_k} \right| < \frac{1}{d_k d_{k+1}}$$

Equipped with this inequality, we can now get to see how good actually approximating an irrational number A by the k-th convergent $\frac{n_k}{d_k}$ actually is.

Theorem. The k-th convergent $\frac{n_k}{d_k}$ provides the *best* rational approximation to A with denominator $\leq d_k$.

Problem 12. Prove this. (Hint: let $N = n_k$, $D = d_k$ and suppose there is a better approximation $\frac{p}{q}$ with $q \leq K$. Try to show first by a triangle inequality[4] that $\left|\frac{p}{q} - \frac{N}{D}\right| < \frac{1}{K^2}$ and at the same time $\left|\frac{p}{q} - \frac{N}{D}\right| \geq \frac{1}{Kq}$).

Let me now conclude the discussion on Continued fractions by a brief mention of semiconvergents and some additional problems. Hopefully, you are by now quite prone to admit the usefulness of continued fractions for all of their nice properties.

[2]Alternating series test
[3]the sum of the first k terms will be bounded in size by the next term
[4]$|x + y| < |x| + |y|$

Definition 5 (Semiconvergents)**.** Consider two consecutive convergents $\frac{n_k}{d_k}$ and $\frac{n_{k+1}}{d_{k+1}}$. Then the fractions of the form

$$\frac{n_k + mn_{k+1}}{d_k + md_{k+1}}$$

for $0 \leq m \leq a_{n+2}$ are called **semiconvergents**.

Theorem. Semiconvergents include all the rational approximations that are better than any approximation with a smaller denominator. Moreover, if $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive semiconvergents, we have $|ad - bc| = 1$.

Problem 13. Find the formula for the continued fraction of ϕ in terms of Fibonacci numbers.

Problem 14. Prove that given an irrational number A, the consequent closest points[5] with integral coordinates to the line $y = Ax$ are in one-to-one correspondence with the convergents of the continued fraction of A.

References

Continued fractions and good approximations.
`http://www.math.jacobs-university.de/timorin/PM/continued_fractions.pdf`

[5] with increasing distance from the origin