Geometry with Complex Numbers

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We all know complex numbers appearing as solutions to polynomial equations and might have seen some other applications of them in Calculus and Analysis.
Surprisingly we usually visualise Complex Numbers in the 2-dimensional plane but we barely use this geometric interpretation. In today's session we are going to see how Complex Numbers can be used to solve geometric problems. It appears not as an intuitive approach and the calculations seem sometimes quite tedious but it turns out that complex numbers are a lot less pain than cartesian coordinates and extremely useful if you are not brilliant in introducing the golden point ;-)
To make sure that everybody starts on a similar level here are some basic properties of complex numbers:

- For every $z \in \mathbb{C}$ we have $z = a + bi = |z|e^{i\theta}$, where θ is the angle between the line from the origin to the complex number and the real axis.

- For all $z_1, z_2 \in \mathbb{C}$ we have

$$z_1 \pm z_2 = (a_1 + a_2) \pm (b_1 + b_2)i,\ z_1 z_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}.$$

- We call $\overline{z} = a - bi$ *complex conjugate* of z and we have:

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2},\ \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2},\ \overline{z_1 \div z_2} = \frac{\overline{z_1}}{\overline{z_2}},\ |z|^2 = z\overline{z}.$$

Now let us investigate which geometric properties complex numbers satisfy.

1. If we scale a complex number z by r we get $z' = rz$.
2. If we translate z by x we get $z + x$.
3. If we rotate a z by an angle θ we get $ze^{i\theta}$.

You might well notice that we are missing the transformation of reflection. This is due to the fact that they are not particulary nice in terms of complex numbers but feel free to find a formula. On the other hand rotations are comparably easy which will be of great usage later. If you feel comfortable with complex numbers and also have seen some geometric interpretation of them feel free to go to Exercise 11 where the fun really starts.

Exercise 1. *Show that any triangle $z_1z_2z_3$ with sidelengths a, b, c and angles α, β, γ in the complex plane satisfies*

$$\left(be^{i\alpha} - c\right) z_1 - be^{i\alpha} z_2 + cz_3 = 0.$$

Exercise 2. *Show that if we fix b, c, α as constants in this equation and insert z_1', z_2', z_3' where $z_1'z_2'z_3'$ is directly similar to $z_1z_2z_3$, then the equation still holds.*

Exercise 3. *Show that for z_1', z_2', z_3' and $z_1'z_2'z_3'$ directly similar to $z_1z_2z_3$ we have $\frac{z_2-z_1}{z_3-z_1} = \frac{z_2'-z_1'}{z_3'-z_1'}$.*

Exercise 4. *What happens to either of those equations if the triangles are not directly similar. How are the two related.*

This is quite a subtle exercise and is it not essential for the further tasks.

Exercise 5. *Show that two line $\overline{AB}, \overline{CD}$ are orthogonal if and only if: $\frac{d-c}{b-a} = -\overline{\left(\frac{d-c}{b-a}\right)}$*

Exercise 6. *Find a similar result for three collinear points A, B, C.*

If we actually want to solve harder exercises in geometry it is important that the calculations we get do not get too long. To achieve this it is first of all important to set the axes in a clever way such that the the coordinates simplify.
In general we will mainly be dealing with triangles and circles but since the latter get complicated quite fast we always try to reduce the task such that only one circle appears which we choose to be the unit circle.

Exercise 7. *Find an expression for the circumcentre, centroid and the orthocentre of a triangle. Hint: How can you place your axes in a clever way?*

Exercise 8. *What can we say about the relation of the three vertices of an equilateral triangle?*

Exercise 9. *What happens to a point on the unit circle if we reflect it on the real axis?*

Exercise 10. *Prove: Four points A, B, C, D lie on a circle if and only if $\frac{(a-c)(b-d)}{(a-d)(b-c)} \in \mathbb{R}$.*

Now let us actually work on some geometric problems. You are very welcome to solve them using Euclidean geometry but I encourage you to try complex numbers and some of them as well to see their power and also their limitations.

Exercise 11. *Let $ABCDEF$ be a hexagon which vertices lie on a circle with radius r. Prove: If $\overline{AB} = \overline{CD} = \overline{EF} = r$, then the midpoints of $\overline{BC} = \overline{DE} = \overline{FA}$ form an equilateral triangle.*

Exercise 12. *Let $ABCD$ be a quadrilateral and $AB \cap CD = P$ and $BC \cap DA = Q$. Prove: the midpoints of $\overline{AC}, \overline{BD}, \overline{PQ}$ lie on one line.*

Exercise 13. *Let ALT, ARM, ORT and ULM four directly similar triangles. Show that A is the midpoint of $\overline{OU}$.*

Exercise 14. *Let A, B, C, D be points on a circle with centre M and let $AC \cap BD = T$. The circumcircles of ATB and CTD intersect a second time in S. Show $ST \perp SM$.*

Exercise 15. *Let $A_1A_2...A_n$ be a regular polygon with radius 1 of the circumcircle. Compute the product of the distances of A_1 to the other vertices of the polygon.*

Exercise 16. *Let $ABCDEF$ be a hexagon which vertices lie on a circle. Let $AB \cap DE = P$, $BC \cap EF = Q$ and $CD \cap FA = R$. Show that P, Q, R are collinear. This is called Pascal's Theorem.*

Exercise 17. *Let $ABCD$ be a quadrilateral with all vertices on a circle. Let H_A, H_B, H_C, H_D denote the orthocentres of BCD, CDA, DAB, ABC. Show that AH_A, BH_B, CH_C, DH_D are concurrent.*

Exercise 18. *Let* $ABCD$ *be a quadrilateral with all vertices on a circle and* P, Q, R *the orthogonal projections of* D *to* BC, CA, AB*. Show that* $\overline{PQ} = \overline{QR}$ *if and only if the bisectors of* $\angle CBA$ *and* $\angle ADC$ *intersect on* AC*.*

Exercise 19. *Let* $ABCDEF$ *be a convex hexagon with all vertices lying on a circle. Let* $\overline{AB} \cdot \overline{CD} \cdot \overline{EF} = \overline{BC} \cdot \overline{DE} \cdot \overline{FA}$*. Show that the diagonals* AD*,* BE *and* CF *intersect in one point.*

Exercise 20. *Let* ABC *be an acute triangle with circumcircle* Γ *and tangent* t *to* Γ*. Let* a, b, c *be lines constructed through reflection of* t *over the lines* BC, CA, AB*. Show that the circumcirlce of the triangle formed by* a, b, c *touches* Γ*.*