

Geometry II.

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Week 9

1 Theory

After having looked into angle-chasing problem solving approaches in geometry last week, the following pages contain simple tools that will enable us to augment angle-chasing with algebraic tools.

More precisely, we will try to reason about angles using lengths and vice versa using Power of a point and Radical lines.

Theorem (Power of a point). *Let s denote distance of point P from center O of a circle k with radius r . The Power of a point with respect to k is a real number h defined as:*

$$h = s^2 - r^2$$

This number reflects the relative distance of P from k .

Furthermore, let an arbitrary line through P intersect k in (not necessarily distinct) points X, Y . Then

$$|PX| \cdot |PY| = h$$

Proof. We will prove only the case where the point lies outside of a circle - other cases can be proved analogically using similar triangles. Consider line through P that intersects circle k in points X, Y and point T also on k such that line PT is tangent to circle k . We know that $\angle PAT = \angle PTB$ (circumscribed angles) and thus by *uu* triangles $\triangle PAT, \triangle PTB$ are similar. This implies that $\frac{|PA|}{|PT|} = \frac{|PT|}{|PB|}$ or equivalently $|PA| \cdot |PB| = |PT|^2$. However, by the Pythagoras theorem also $|PT|^2 = s^2 - r^2$. \square

Exercise 1. *Prove that Power of a point theorem holds for any n -dimensional sphere, $n \geq 2$.*

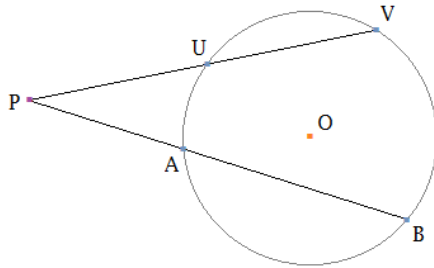


Figure 1: Here, $|PA| \cdot |PB| = |PU| \cdot |PV| = h$

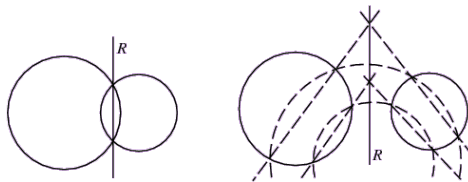
Theorem. *Two circles are orthogonal if they intersect in right angles. Equivalently:*

$$r_1^2 + r_2^2 = d^2$$

where r_1, r_2 are radii of the circles and d distance of their centers.

Theorem (Chordal theorem). *Given two circles k, l with distinct centers, the set of points such that their power is same with respect to k and with respect to l is a line $R_{k,l}$ perpendicular to the line joining centers of k, l . It is known as radical axis of the two circles.*

Furthermore, the radical axis is set of centers of all circles that are orthogonal to both k, l .

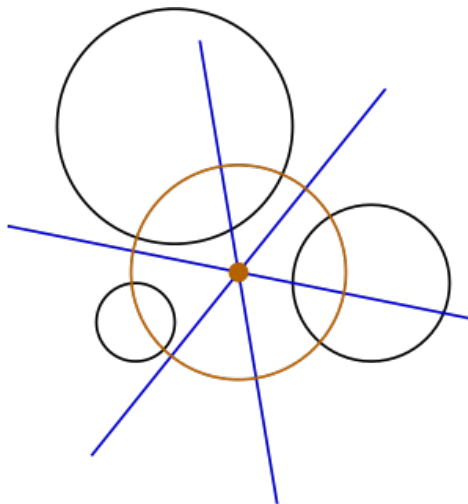


Exercise 2. *Considering the definition of orthogonal circles above, prove that a radical axis is indeed a set of centers of all circles that are orthogonal to k, l .*

Theorem (Radical axis theorem). *Let there three circles a, b, c given such that no two are concentric. The three radical axes (for each pair of circles) intersect in a single point called the radical center, or are parallel.*

This also means that there is a unique circle with its center at the radical center that is orthogonal to all three circles.

Proof. Consider radical axes $R_{a,b}, R_{b,c}$ intersecting in point C . By definition of a radical axis the tangents to a and b through C are equal in length, as well as tangents to b and c through C are. By the transitivity of equality, also tangents to a and c through C are equal in length, which means that C also lies on $R_{a,c}$. \square



Exercise 3. Prove by similar argument that the unique circle orthogonal to all three circles is indeed centered in radical center.

Let's finish the theory section by mentioning that radical lines play important role in solution of a Problem of Apollonius ¹.

2 Problems

Easy

1. Prove that quadrilateral $ABCD$ with $M = AC \cap BD$ is cyclic if and only if $|MA| \cdot |MC| = |MB| \cdot |MD|$.

¹Given three objects, each of which may be a point, line, or circle, draw a circle that is tangent to each

2. Given line segments of lengths x and y , construct a line segment of length \sqrt{xy} .
3. We are given line p and points A and B on the same side of p but not on p . How can we construct circle tangent to p through points A and B ?
4. We're given two circles and which have a common tangent. Let the tangent points be A and B . Show that chordal of the two circles divides AB in half.
5. Prove that heights of a triangle intersect in single point.

Medium

6. We are given line p and points A and B on the opposite sides of p . Find circle through A and B , such that the length of p inside the circle is minimal.
7. Let $ABCD$ be a quadrilateral with $AB \parallel CD$, $AB > CD$, $AC \perp BD$. Let O be circumcenter of $\triangle ABC$ and $E = OB \cap CD$. Show that $BC^2 = CD \cdot CE$.
8. On lines CA and CB of acute triangle ABC there are points X and Y . Let P be intersection of AX and BY and Q be intersection of circumcircles of triangles AXC and BYC (other than C). Prove that C , P , and Q are on a line if and only if $ABXY$ is cyclic.

Difficult

9. ABC is an acute triangle and H its orthocenter. Circle with diameter AH intersects circumcircle in points A and K . Line KV intersects BC in M . Prove that M is midpoint of BC .
10. Let circles k and l be externally tangent at point T . Let P be any point on l . Tangent lines to k through P intersect k at A and B . Let $AT \cap l = C$ and $BT \cap l = D$. Let t be tangent to l at P and $M = t \cap CD$. Find set of all possible positions of M as we move P along l .

11. On lines AB and AC of acute triangle ABC there are points M and N . Circles with diameters BN , CM intersect at P and Q . Prove that P , Q , and orthocenter H are colinear.

References

- [1] Ondrej Budáč, Tomáš Jurík, and Ján Mazák. *Zbierka úloh KMS*. Trojsten, Bratislava, 2010.
- [2] Matematický korespondenční seminář. Knihovna. [ONLINE] Available at: <https://mks.mff.cuni.cz/library/library.php>. [Accessed November 10].