Geometry I.

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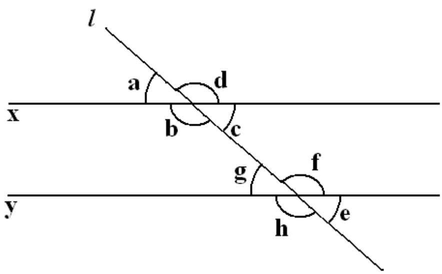
Week 8

1 Theory

One of the most beautiful (I might be biased here) topics in problem solving is Euclidean geometry. To solve a geometric problem you often need to have a cunning perspective over the whole problem and employ creativity.
In this sheet we will briefly iterate over some well-known theorems and look in more detail into properties of cyclic quadrilaterals.
Let us adapt the notation $\alpha = \beta$, which denotes that angles α, β are *congruent* (equal in measure).

Basic theorems

Theorem (Parallel lines)**.** *In the figure below, the lines x and y are parallel. Then we have $a = c = g = e$, $b = d = f = h$, and $b + g = 180°$.*



Pair of angles a, e is an example of *alternate exterior angles.*
Pair of angles c, g is an example of *alternate interior angles.*
Pair of angles a, c is an example of *vertical angles.*

Exercise 1. *In the figure 1, supposte that* $a = 3x - 33°$ *and* $h = 7x + 3°$. *Find* x.

Theorem (Similarity of triangles)**.** *Two triangles are similar if and only if corresponding angles are congruent and the lengths of corresponding sides are proportional.*
Furthermore, two triangles are similar if:

- ***aa****: the triangles have two congruent angles*
- ***sss****: all corresponding sides have lengths in the same ratio*
- ***sas****: two sides have lengths in the same ratio, and the angles included between these sides are congruent*

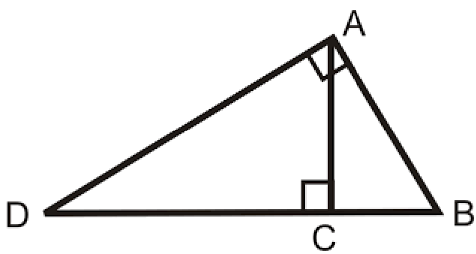
Exercise 2. *Prove aa, sss and sas conditions above.*

Theorem (Pythagoras theorem)**.** *In arbitrary right-angled triangle (figure below)*

$$|AB|^2 + |AD|^2 = |BD|^2$$

Theorem (Thales' theorem)**.** BD *is diameter of circumcircle of* $\triangle ABD$.

Theorem (Geometric mean theorem)**.**
Altitude Rule: $|CA|^2 = |CD| \cdot |CB|$.
Leg Rule: $|BA|^2 = |BC| \cdot |BD|$.

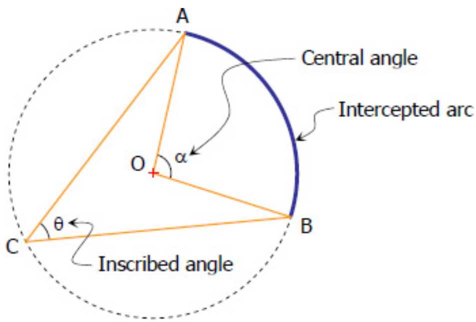


Exercise 3. *Prove* Geometric mean theorem *(both rules).*

Theorem (Central angle theorem). *Let AB be arbitrary arc of a circle with center O and C arbitrary point lying on a circle, but not on arc AB. Then*

$$\angle AOB = 2\angle ACB$$

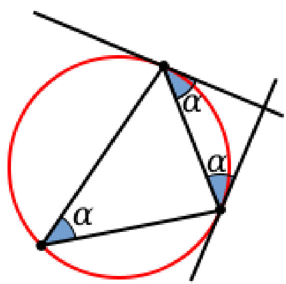
Proof. We will prove the case when O lies between segments BC and AC. Denote $\angle ACO$ as x and $\angle BCO$ as y. Since $\triangle AOC$ is isosceles triangle, $\angle CAO = x$ and $\angle AOC = 180° - 2x$. Analogically, $\angle BOC = 180° - 2y$. Since angles $\angle AOC$, $\angle BOC$ and $\angle AOB$ must add up to $360°$, we must have $\angle AOB = 2x + 2y$.
In the figure above, $\alpha = 2x + 2y = 2(x + y) = 2\theta$. □



Exercise 4. *Prove the general case of Central angle theorem.*

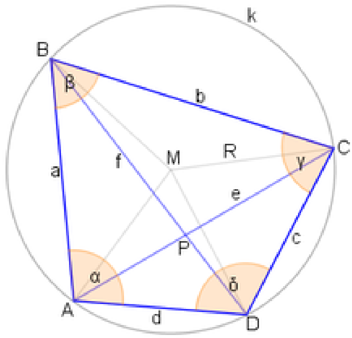
Theorem (Circumscribed angles). *Let AB be arc of a circle centered in O (such that length of this arc spans at most half of the circle) with corresponding inscibed angle α. Then the angles between the tangent lines to a circle in points A, B and the line segment AB is congruent to α.*

Proof. Since $\triangle AOB$ is isosceles and by the central angle theorem $\angle AOB = 2\alpha$, we must have $\angle OAB = \angle OBA = 90° - \alpha$. Combining this with the fact that segment OA is perpendicular to the tangent line at point A, we obtain the sought result. □



Cyclic quadrilaterals

Many geometric problems are solvable only by seeking cyclic quadrilaterals in them and applying their properties. Let us take a good look at them:



Theorem (Properties of cyclic quadrilaterals)**.** *The following statements are all equivalent:*
- $ABCD$ is cyclic quadrilateral
- Points A, B, C, D all lie on a common circle
- Each exterior angle of $ABCD$ is equal to the opposite interior angle.
- The four perpendicular bisectors to the sides are concurrent. Their common point of intersection is the circumcenter.
- Angle between any side and one diagonal is equal to the angle between opposite side and the other diagonal. That is, for instance, $\angle ABD = \angle ACD$
- The opposite angles are supplementary[1], that is: $\alpha + \gamma = \beta + \delta = 180°$

Exercise 5. *Try to find the reasons for why the above properties hold, using central angle theorem.*

We will conclude the theory section by looking at two more interesting properties of cyclic quadrilaterals:

Theorem (Ptolemy's theorem)**.** *Let e, f denote the lenghts of diagonals of a cyclic quadrilateral $ABCD$ with lengths of the sides a, b, c, d. Then*

$$ef = ac + bd$$

[1] they add up to 180°

Theorem (Brahmagupta's formula)**.** *The area A of a cyclic quadrilateral can be found as*

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = \frac{a+b+c+d}{2}$ is the semiperimeter.

2 Problems

Easy

1. Let $ABCD$ be a parallelogram, W, X, Y, Z points on the sides AB, BC, CD, DA and S intersection of XZ with WY. $AWSZ$ is cyclic. Prove that also $BXSW, XCYS, YDZS$ are cyclic.

2. We are given triangle ABC. Angle bisector of $\angle BCA$ intersect circumcircle of $\triangle ABC$ in point Š. Prove that Š is midpoint of arc AB.

3. Circles l and k intersect in points A, B. Choose point X on the circle k and let Y be intersection of l with line XB. Show that $\angle AXY$ does not depend on choice of X

Medium

4. Given triangle ABC, let H be intersection of its altitudes (orthocenter). Prove that reflection of H over line AB lies on the circumcircle of $\triangle ABC$

5. Let the points of intersection of the altitudes with the sides of the triangle $\triangle ABC$ be D, E, and F. Show that altitudes of $\triangle ABC$ are angle bisectors in $\triangle DEF$.

6. Let ABC be a triangle with right angle at C and M be a point on line segment AB. Let S, S_1, and S_2 be circumcentres of triangles ABC, AMC, BMC Show that M, C, S, S_1, and S_2 lie on a circle.

7. Given triangle ABC and a point D on its circumcircle, let P, Q, R be points closest to D on sides AB, BC, CA. Prove that P, Q, R colinear.

Difficult

8. Points A, B, C, D, E, F lie on a circle. Let AE and BF intersect at P, BD and CE at R, and AD and CF at S. Show that P, R, S are colinear.

9. Let O be circumcenter of triangle ABC. Line through O intersect sides AB and AC in M and N. Let R, S be midpoints of CM, BN. Show that $\angle ROS = \angle BAC$

References

[1] Ondrej Budáč, Tomáš Jurík, and Ján Mazák. *Zbierka úloh KMS*. Trojsten, Bratislava, 2010.

[2] Matematický korespondenční seminář. Knihovna. [ONLINE] Available at: `https://mks.mff.cuni.cz/library/library.php`. [Accessed November 10].