Problem Solving Maths Group
Lines and Circles in Combinatorics

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In this session, we will look at two basic geometrical structures, circles and lines and how they interact in the plane. We will count the intersections between n lines and k circles and the number of regions they divide the plane into. Try to search for patterns and justify your findings. No specific theorems are needed, but analytic geometry and combinations may be useful.

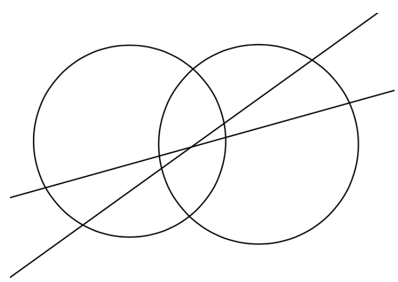

Figure 1: For $n = k = 2$ the number of intersections is 11.

Problem 1 - Intersection of n lines

Consider two lines. They either intersect each other at one point, are parallel or coincide. The number of maximum intersections of two lines is one.
How many are the intersections of 3 lines, 4 lines, or $\ldots$ n lines? Complete the table and guess a formula for the maximum number of intersections. Try to prove your formula is correct.

Problem 2 - Regions n lines divide the plane into

One line intersects the plane into two regions. How many regions can n lines divide the plane into?

Problem 4 - The pizza slicing problem

You and four friends have a pizza party. How many slices can you make if each person cut once? How does this problem relate to the preceding one?

Number of lines	Number of intersections
0	
1	
2	
3	
4	
5	

Number of lines	Number of regions
0	
1	
2	
3	
4	
5	

Problem 3 - Intersections and regions of the plane of n circles
We now move on to circles. Try to complete the table and come up with a formula for the number of intersections of n circles and the number of regions they divide the plane into.

Number of circles	Number of intersections	Number of regions
0		
1		
2		
3		
4		
5		

Problem 4 - Intersections and regions of the plane of n circles and k lines
We now tie lines and circles together. What is maximum number of intersections of n circles and k lines? How many regions do they divide the plane into?

Problem 5 - Analysis
Compare the formulas for the number of intersections and regions. Which one grows faster? What would you expect? What is the rate of growth, linear,

quadratic, exponential, ...? Given the same number of lines and circles, what object would yield more intersections and regions?

Challenge - 3 dimensions
Carry out the investigation for planes and spheres!

We will now look at a definition of a halving life and circle and try to count them.

Definition 1: Assume there are n points in a plane, n even, where no three points are collinear. A line is a *halving line* if it goes through two points and has $\frac{n-2}{2}$ points on either side.

Problem 6. Prove that such a line always exists.
Problem 7. Prove there are in fact at least $n/2$ such lines.

Definition 2: Assume there are n points in a plane, n odd, where no three points are collinear and no four on the same circle. A circle is a *halving circle* if it goes through three points and has $\frac{n-3}{2}$ points both in its exterior and interior.

Problem 8. Prove that there is always such a circle.
Problem 9. Show that for any two points it is guaranteed to exist at least one circle going through them. Use that to prove that there are at least $\frac{n(2n+1)}{3}$ halving circles.

Problem 10 - Super Challenge: Gauss's Circle Problem
Consider a coordinate system with points on every integer coordinate. How many such points $N(r)$ is contained in a circle of radius r centred at the origin? [An explicit formula was not found until 1999!]
What should this answer be asympotically? Can you obtain an upper and lower bound?
Prove that

$$N(r) = \sum_{n=0}^{n^2} r(n)$$

where $r(n)$ is the number of ways you can write n as a sum of two squares.