Mean Inequalities

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Week 6

1 Theory

Basics

Inequality problems are a very broad topic. In this text, we will mostly focus the inequalities between means and how to apply them to solve wide range of inequalities.

We will be only concerned with real numbers (often only non-negative real numbers). One of basic properties of real numbers is that the square of a real number is non-negative, thus so is any sum of squares. Here follows the most simple but useful inequality.

Theorem (Sum of squares inequality). *If* $x_1, x_2, \ldots, x_n \in \mathbb{R}$ *and* $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{R}$, *with* $\alpha_1, \alpha_2, \ldots \alpha_n > 0$, *then* $\sum_{i=1}^{n} \alpha_i x_i^2 \geq 0$ *with equality if and only if* $x_1 = x_2 = \cdots = x_n = 0$.

Example 1. *Prove that* $x^2 + y^2 + 2 \geq 2x + 2y$ *and find when there is equality.*

Proof.

$$x^2 + y^2 + 2 \geq 2x + 2y$$
$$x^2 - 2x + 1 + y^2 - 2y + 1 \geq 0$$
$$(x-1)^2 + (y-1)^2 \geq 0$$

So our inequality is equivalent to the $(x-1)^2 + (y-1)^2 \geq 0$, which is Sum of squares inequality with equality for $x = y = 1$ □

Exercise 1. *Prove that* $4x^2 + y^2 + 3 \geq 4x + 2y$.

Thanks to the work of Emil Artin and Charles Delzell it is now known, that it is possible to prove any inequality involving rational functions simply by reducing it to a sum of squares. However, to do so would be outstandingly tedious and so we introduce much more powerful method - Arithmetic-Geometric mean inequality.

Arithmetic-Geometric mean inequality

A well known inequality, often used in problem solving, is an inequality between arithmetic and geometric mean. It states that geometric mean of any nonnegative real numbers is less than or equal to arithmetic mean of these numbers. It is often called AM-GM inequality or simply AG.

Theorem (AG)**.** *For nonegative real numbers* $x_1, x_2, \dots x_n$*:*

$$\sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

We will prove AG for $n = 2$. The AG becomes $\sqrt{x_1 x_2} \leq \frac{x_1+x_2}{2}$.

Proof. Using that square of a real is non-negative and basic algebraic operations:

$$\begin{aligned} 0 &\leq (x-y)^2 \\ &= x^2 - 2xy + y^2 \\ &= x^2 + 2xy - y^2 - 4xy \\ &= (x+y)^2 - 4xy \end{aligned}$$

Thus $4xy \leq (x+y)^2$ from which $\sqrt{x_1 x_2} \leq \frac{x_1+x_2}{2}$ follows imidiately. □

Example 2. *Prove that for possitive real numbers* x, y, z*:* $x^3 + y^3 + z^3 \geq x^2y + y^2z + z^2x$.

Proof. Let's look at each expression on the right-hand side separately. We can treat x^2y as product of three numbers $x \cdot x \cdot y$ and using AG we have

$$x^2y = x \cdot x \cdot y = \sqrt[3]{x^3x^3y^3} \leq \frac{x^3 + x^3 + y^3}{3}$$

We obtain analogical inequalities for y^2z and z^2x - summing those three up we get exactly the inequality we wanted to prove. $\square$

Exercise 2. *Prove that for possitive real numbers x, y, z: $x^4 + y^4 + z^4 \geq x^2yz + xy^2z + xyz^2$.*

Weights and Power Mean Inequality

There are some generalizations of AM-GM inequality, first of them uses so called weighted means.

Theorem (Weighted AG)**.** *For nonegative real numbers $x_1, x_2, \dots x_n$ and non-negative real weights $w_1, w_2, \dots w_n$ where $w = w_1 + w_2 + \cdots + w_n > 0$, the following holds:*

$$\sqrt[w]{x_1^{w_1} x_2^{w_2} \dots x_n^{w_n}} \leq \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w}$$

with equality if and only if all the x_k with $w_k > 0$ are equal.
Notice that the original AG inequality is just a special case with all $w_i = 1$.

Example 3. *Prove that for possitive real x, y, z: $\frac{x^2}{2} + \frac{y^3}{3} + \frac{z^6}{6} \geq xyz$.*

Proof. Notice that $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ Using weights $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$, the weighted AG for numbers x^2, y^3, and z^6 gives us exactly the inequality we wanted to prove. $\square$

Exercise 3. *Prove that for possitive real x, y, z: $\frac{1}{2x^2} + \frac{1}{3y^3} + \frac{1}{6z^6} \geq \frac{1}{xyz}$.*

Another very powerful generalization arises when looking at other means, not just arithmetic or geometric. The generalized mean, or power mean, is defined as follows:

Theorem. AG. *If p is a non-zero real number and $x_1, x_2, \dots, x_n$ are positive integers then power mean with exponent p is*

$$M_p(x_1, x_2, \dots, x_n) = \left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}}$$

We can see that for $p = 1$, M_1 is arithmetic mean.
For $p = 0$, the M_p is not defined, however, it can be shown that

$$\lim_{p \to 0} M_p(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \dots x_n}$$

The geometric mean is therefore usually assigned to be M_0. So far we have discussed $M_0 \leq M_1$. So how do other means relate to this?

Theorem (Generalized mean inequality.). *For any positive real numbers* $x_1, \ldots, x_n$ *if* $p < q$ *then* $M_p(x_1, \ldots, x_n) < M_q(x_1, \ldots, x_n)$.

As a final note concerning terminology, the M_{-1} is usually called Harmonic mean and M_2 quadratic mean.

2 Problems

Easy

1. Demonstrate that if $a_1 a_2 \cdots a_n = 1$ then $a_1 + a_2 + \cdots + a_n \geq n$.
2. Demonstrate that $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq 3$ for any positive numbers x, y, z.
3. For nonnegative real numbers x, y, prove that $2x^3 + y^3 \geq 3x^2 y$.
4. For positive real number x, prove that $x^2 + \frac{2}{x} \geq 3$.
5. Show that for positive real numbers x, y, z:
   $x^3(x + 2y) + y^3(y + 2x) \geq 6x^2 y^2$.

Medium

6. Show that if for real positive numbers a, b we have $a + b = 1$, then also $a^3 + b^3 \geq ab$.
7. For $x, y, z \in \mathbb{R}^+$ prove $x^3 y + y^3 z + z^3 x \geq x^2 yz + y^2 zx + z^2 xy$.
8. Let x, y, z be real positive numbers for which we have
   $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 2$. Show that $8xyz \geq 1$.

Difficult

9. For positive real numbers a, b, c, show that $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c$.
10. Show that for any positive real numbers a, b, c with $abc = 1$ the following inequality holds: $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c$

11. Let a, b, c be positive real numbers for which $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt{3}+3}{2}$$

Extra

12. Prove AG using induction.

13. Prove that power mean tends to geometric mean as the power tends to zero.

References

[1] Ondrej Budáč, Tomáš Jurík, and Ján Mazák. *Zbierka úloh KMS*. Trojsten, Bratislava, 2010.

[2] Matematický korespondenční seminář. Knihovna. [ONLINE] Available at: `https://mks.mff.cuni.cz/library/library.php`. [Accessed 10 October 26].

3 Hints

Easy

1. Use basic AG.

2. Use AG directly.

3. Divide left side into three terms carefully and use basic AG.

4. Divide left side into three terms carefully and use basic AG.

5. As the number on the right-hand side suggests, try to divide the left side into 6 terms.

Medium

6. Factor left side.

7. Notice similarity with Example 2.

8. Rewrite so that there are no fractions. Look for convenient terms for which arithmetic or geometric mean is in the inequality that you want to prove.

Difficult

9. What can you say about $\frac{a^2}{b} + b$?

10. Find and sum more AG inequalities conveniently.

11. -

Extra

12. Use the three step induction. Show that if AG holds for n numbers then it also holds for $2n$ numbers and then that if it holds for m numbers it also holds for $2m - 1$ numbers. (Why this induction works?)

13. Use the L'Hôpital's rule.