Problem Solving Challenge 2018

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Introduction

Why problem solving?

Our modern world is filled with problems of all kinds, ranging in scale, complexity and importance. One of the crucial factors that has contributed to the success of human development overall is our ability to solve or overcome these problems, using our intelligence and strength in numbers. Hence, there will always be a need for individuals who are keen problem solvers - if not for saving the world, perhaps for saving yourself some trouble. But as with all things, becoming a good problem solver requires three things: Practice, practice and practice.

Why maths?

Today's event is about mathematics, which at its core, is concerned with solving problems of logical nature, using tools of logic. Although not all problems are of logical nature, practising problem solving in the context of mathematics makes you practice general problem solving skills, such as:

- Coming up with creative ideas;
- Testing your ideas and getting used to failure;
- Mental stamina;
- Collaborating with other people.

At the end of the day, however, what matters the most is your *attitude* towards a problem. Will you do what it takes to solve it, or will you give up to presuppositions in your head? This is particularly important when you do mathematics, which can be very abstract and hard to get your head around at first. We really encourage you not to give up - you are often capable of more than you think!

How do I solve mathematical problems?

You will all have had different exposure to and experiences with mathematics, and some problems may be easier or harder to understand depending on your mathematical background - don't worry, we the organisers are always more than happy to try to help you whatever your struggle is. However, here are some general points to be made.

- Make sure you understand the question. This may seem obvious, but often it is halfway to solving the problem. What do all the terms mean, and what does the problem require you to do? What facts do I have at my disposal, and what result do I want to end up with? It's a lot like doing puzzles - you look at the pieces and try to figure out how to construct the beautiful picture at the end. Sometimes drawing figures also helps!
- Sometimes mathematics asks you to *prove* something. Proving something is essentially showing that something is true beyond all reasonable doubt. For example, I could get the question: "Let $x = 2^3 + 5$. Prove that $x = 13$." Using the laws of arithmetic, I have that

$$\begin{aligned} x = 2^3 + 5 &\implies x = 8 + 5 \quad \text{(using definition of exponents and multiplication)},\\ &\implies x = 13 \quad \text{(using addition)}. \end{aligned}$$

  You have used the laws of mathematics, which you know are true, to prove something.

- Ask a lot of questions! Although it is great to practice your endurance, you will not gain anything by sitting clueless at a table for several hours. You will be sitting in groups, so make sure to discuss things with your peers. What are their thoughts on the problem and what angle have they attacked it from? Sometimes solutions to problems lie at the intersection of ideas.
- Take a break! Whether or not this means eating a sandwich or trying to solve a different problem, sometimes the greatest ideas come to you when walking away from the problem for a while.

Great, where do I start?

Below is an introduction to the theme of this year's challenge: probability and expectation. After the introduction there are three problems to choose from - Buffon's needle, Geobard shapes, and St. Petersburg's Paradox. All of them uses expectation in some way. You are free to choose which problem you want to do, and you are also free to jump between them. As the problems are pretty difficult to solve on their own, we have provided suggestions in order to guide you through the problems - these can be difficult in themselves, but lead to nice solutions! Having that said, there are always other ways to solve the problem, so if you find one, that's awesome.

You are not required to write your solutions down or anything, as long as you can convince yourself that you understand what is going on. We the organisers will, however, out of curiosity and feedback purposes, ask you at the end of the day what you have learned, what results you have come up with, and what you have found interesting about the challenge.

Mathematics is a fantastic subject on its own, and some of the results in today's challenge are surprising and rather extraordinary. We hope that this can fuel your curiosity in wanting to solve as many of the problems as possible! Finally, there is not much like the feeling you get when you have wrestled with a difficult problem and you finally solve it - and it doesn't matter what "level" it is on. What is important is that you are challenged and that you enjoy it.

So without further adue, on your marks, get set, go!

The power of linearity

Expected value [Expectation]

Definition (Random variable)**.** We will, simplistically, define a *random variable* to be a variable whose possible values are outcomes of a random phenomenon.

Definition. A probability event, or an simply *event*, is a subset E of the set Ω of all possible outcomes. We say that the probability of the event E occurring is the real number $\frac{|E|}{|\Omega|}$ and $0 \leq \frac{|E|}{|\Omega|} \leq 1$.

An example of a random variable may be the number obtained when a die is rolled. The set of possible outcomes is $\{1, 2, 3, 4, 5, 6\}$, where each outcome is associated with its probability (in this case each number has probability[1] equal to $\frac{1}{6}$, as there are 6 equally likely outcomes). This assignment of probabilities to outcomes is usually referred to as a *probability distribution*. When the random variable is clear from the context, we will write p_k to denote the probability of a possible outcome x_k. The set of all possible outcomes is often denoted as Ω (the *sample space*).

Example. Let X be a random variable denoting the sum of two independent dice rolls. Then the set of possible outcomes is:

$$x_1 = 2,\ x_2 = 3,\ \ldots,\ x_{11} = 12$$

with corresponding probabilities:

$$p_1 = p_{11} = \frac{1}{36},\ p_2 = p_{10} = \frac{2}{36},\ p_3 = p_9 = \frac{3}{36},\ p_4 = p_8 = \frac{4}{36},\ p_5 = p_7 = \frac{5}{36},\ p_6 = \frac{6}{36}$$

Notice that we may think about X as being a sum of two other random variables, $X = X_1 + X_2$, where X_1, X_2 are the random variables denoting the value on the first and the second die respectively.

Suggestion 1. What is the set of all possible sums of three dice? What are the corresponding probabilities? What about n dice?

Definition (Expectation)**.** The *expected value* of a random variable X with possible outcomes $x_1, x_2, \ldots, x_k$ is:

$$\mathbb{E}(X) := x_1 p_1 + x_2 p_2 + \cdots + x_k p_k$$

In more compact notation, this can be also written as:

$$\mathbb{E}(X) = \sum_{x \in \Omega} x \cdot p(x)$$

The expected value may be intuitively thought about as being some sort of an *average outcome*.

Example. The expected value of a roll of a die is $\frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$.

Theorem (The linearity of expectation)**.** Let X, Y be *arbitrary*[2] random variables. Then:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$

More generally, the following holds (where c_i are any real numbers):

$$\mathbb{E}\left[\sum_{i=1}^{n} c_i X_i\right] = \sum_{i=1}^{n} c_i\, \mathbb{E}(X_i)$$

Example. The expected value of the sum of two independent dice rolls X is:

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 3.5 + 3.5 = 7$$

Suggestion 2. What is the expected value of the sum of n fair k-sided dice rolls?

Definition (Indicator variables)**.** We will call an *indicator variable* any random variables that only takes the values 0 or 1, denoting the absence or presence of some property.

[1] Assuming a fair die.
[2] They may as well be dependent on each other.

Example. A die is rolled. Consider the indicator variables R_i:

$$R_i = \begin{cases} 1 & \text{the number } i \text{ was rolled} \\ 0 & \text{otherwise} \end{cases}$$

We have $\mathbb{E}(R_i) = 1 \cdot \frac{1}{6} + 0 \cdot \frac{5}{6} = \frac{1}{6}$. The value R that was rolled can then be written as $R = \sum_{i=1}^{6} iR_i$, and, by the linearity of expectation we find, as expected:

$$\mathbb{E}(R) = \mathbb{E}\left[\sum_{i=1}^{6} iR_i\right] = \sum_{i=1}^{6} i\,\mathbb{E}(R_i) = \frac{1}{6}\sum_{i=1}^{6} i = 3.5$$

Although the indicator variables are a trivial notion, their correct use may simplify otherwise difficult problems substantially.

Example (The hat-check problem)**.** On a sunny day, n men enter a fancy Italian restaurant and leave all of their n hats at the reception. As soon as all of the pizza is eaten, the group leaves in hurry - the men take their hats from the reception at random. What is the expected number of people that have entered and left the restaurant with the same hat?

Let R_i be an indicator variable defined as:

$$R_i = \begin{cases} 1 & \text{the } i\text{-th person got his hat back} \\ 0 & \text{otherwise} \end{cases}$$

The probability of one particular person getting his hat back is $\frac{1}{n}$, as there are n hats. Hence

$$\mathbb{E}(R_i) = 1 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

The number R of people leaving with their own hat will then be just $\sum_{i=1}^{n} R_i$ and hence, by the linearity of expectation:

$$\mathbb{E}(R) = \mathbb{E}\left[\sum_{i=1}^{n} R_i\right] = \sum_{i=1}^{n} \mathbb{E}(R_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Suggestion 3 (Practice on indicator variables)**.** Suppose that a team of people is choosing at random exactly 4 suggestions that they will go through. Available are 6 theoretical ones and 3 computational ones (mutually exclusive). Let C be the number of computational ones chosen. Use indicator variables to find $\mathbb{E}(C)$.

Buffon's needle

In the 18-th century, Georges-Louis Leclerc, Comte de Buffon, a French nobleman, had a curious question to ask. His question was one of the earliest problems in geometric probability and it was indeed curious, as there will be π involved. The following sequence of suggestions will lead us to the answer to his curious problem - and beyond.

Problem (Buffon's needle)**.** Suppose that you drop your favourite needle (it is your favourite one because it has zero width) of length l onto an infinite ruled paper which has the distance between the lines d. What is the probability p that your needle crossed at least one of the lines?

Towards the answer

It may come as a surprise - but the answer to the above problem will let any poor mathematician, equipped with just a needle and a sheet of ruled paper, estimate π, with quite a remarkable accuracy.

Suggestion 4. Come up with an expression for the expected number of crossings, denoted E_c (in terms of probabilities - you are not expected to evaluate it).

Suggestion 5. What is the probability that the needle will be lined-up with one of the lines?

Suggestion 6. Let p_i denote the probability that the needle will have i crossings. What is the sought probability p in terms of p_i's?

Suggestion 7. Under what conditions on l and d can we equate $p = E_c$?

From now on, we will consider **only the case where the above condition holds**. Define $\mathbb{E}_c(l)$ to be the expected number of crossing when dropping a needle of length l and $p(l)$ the probability of the needle of length l crossing one of the lines when dropped. The linearity of expectation gives us for any $x, y \geq 0, x + y = l$:

$$p(l) = \mathbb{E}_c(l) = \mathbb{E}_c(x) + \mathbb{E}_c(y) = p(x) + p(y)$$

Suggestion 8. Use the above equation to show the following. Clarify what "suitably small x" is in each case, referring to the condition from Suggestion 8.

- $\mathbb{E}_c(nx) = n\,\mathbb{E}_c(x)$ for all $n \in \mathbb{N}$ and suitably small $x \in \mathbb{R}$
- $\mathbb{E}_c(qx) = q\,\mathbb{E}_c(x)$ for all $q \in \mathbb{Q}$ and suitably small $x \in \mathbb{R}$
- Now try to justify that $\mathbb{E}_c(rx) = r\,\mathbb{E}_c(x)$ for all $r \in \mathbb{R}$ and suitably small $x \in \mathbb{R}$
- $\mathbb{E}_c(x) = \mathbb{E}_c(1)x$ for suitably small $x \in \mathbb{R}$

Thus to determine $\mathbb{E}_c(x)$, it suffices to find out what $\mathbb{E}_c(1)$ is.

Suggestion 9. Show that if we drop instead of the needle any shape consisting of some number of straight edges of total length l, the expected number of crossings is (still) $\mathbb{E}_c(1)l$

Consider now a circle $\mathcal{C}^{(\infty)}$ of radius d.

Suggestion 10. What is the expected number of crossings when the circle $\mathcal{C}^{(\infty)}$ is dropped?

Consider moreover two regular n-gons: $\mathcal{C}_i^{(n)}$ inscribed to $\mathcal{C}^{(\infty)}$, and $\mathcal{C}_c^{(n)}$ circumscribed to $\mathcal{C}^{(\infty)}$, with perimeters $\mathcal{P}_i$ and $\mathcal{P}_c$ respectively.

Suggestion 11. Show that $\mathcal{P}_i\,\mathbb{E}_c(1) \leq 2 \leq \mathcal{P}_c\,\mathbb{E}_c(1)$.

Suggestion 12. Deduce, finally, the value of $\mathbb{E}_c(1)$.

Suggestion 13. How can the above result be used for a manual estimation of π? If you can, write a computer program that estimates π by this procedure. How does your relationship between the number of drops and the error in the estimation look like?

And beyond...

Suggestion 14. What goes wrong when the condition from Suggestion 8 does not hold?

Suggestion 15. In the exploration, we encountered circles and collections of line segments being dropped. Are there generalizations to other shapes[3]?

Suggestion 16. Can you formulate a similar problem in 3 dimensions? Can it be solved analogously?

Suggestion 17. The problem can be solved purely by integral calculus. Try to come up with such (or perhaps another) solution.

Geoboard shapes

In the following, we will explore an interesting problem on a *geoboard*. Geoboards are educative tools for "playing around" with concepts such as perimeter, area and other characteristics of polygons. They were invented and popularized by Egyptian mathematician Caleb Gattegno about 50 years ago.

Problem (Geoboard shapes[4])**.** A **geoboard** of order n is an $n \times n$ square board with a hole at each of the $(n+1)^2$ lattice points on the square.
Simon, who has an infinite amount of pins and mathematical curiosity, approaches the board and for each of the $(n+1)^2$ holes does the following:
→ Rolls an $(n+1)$-sided die.
→ If the number 1 was rolled, he places a pin in the particular hole.
After the procedure was done for each hole, he wraps a tight rubber-band around all of the pins that have been placed onto the board - forming a shape S. Let $A_n(S)$ be the area of the shape S, provided that Simon had a geoboard of order n. What is $\mathbb{E}(A_1)$? $\mathbb{E}(A_2)$? $\mathbb{E}(A_{50})$?

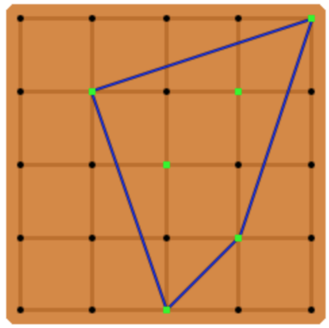

Figure 1: Example for a geoboard of order $n = 4$. The green markers denote the placed pins and the blue line denotes the rubber-band. In this particular arrangement, the shape S has area 6.

For reference, the shape formed by the tight rubber is known as the *convex hull* of a set of points (in our case the pins). Even though a particular task has been specified for the problem, it still may be useful to perform an analysis from various points of view.

Suggestion 18. What is the expected number of pins on the geoboard after the procedure, as a function of n?

Suggestion 19. What is the expected number of distinct unordered *pairs* of pins[5] on the board after the procedure, as a function of n?

One may notice that there exist cases when the area of the shape will turn out to be 0 or n^2. What are these cases?

Suggestion 20. Pick a geoboard of a small order (say $n = 2$ or $n = 3$). What is the probability that $A_n(S) = 0$? What about a general n?

Suggestion 21. What about the probability that $A_n(S) = n^2$?

Suggestion 22. Can you find any lower or upper bounds on $\mathbb{E}(A_n)$?

One issue one may encounter conceptually is how to relate the positions of the vertex-pins of S to $A_n(S)$. We will give (as an example) two ways on how to do this.

Theorem (Shoelace theorem)**.** Let P be a polygon with vertices $V_1 = (x_1, y_1), V_2 = (x_2, y_2), \ldots, V_k = (x_k, y_k)$ *listed in counterclockwise order.* Then the area A of P is:

$$\begin{aligned} A &= \frac{1}{2}\left[(x_1y_2 + x_2y_3 + \cdots + x_ky_1) - (y_1x_2 + y_2x_3 + \cdots + y_kx_1)\right] \\ &= \frac{1}{2}(V_1 \times V_2 + V_2 \times V_3 + \cdots + V_k \times V_1) \end{aligned}$$

where $\times$ denotes the cross product - that is, for pairs (a, b) and (c, d) we have that

$$(a, b) \times (c, d) = ad - bc.$$

Suggestion 23. Apply the shoelace theorem on the geoboard shape in the figure above.

Suggestion 24. Prove the Shoelace theorem for the case where P is a triangle. Use this to prove the Shoelace theorem for general P. Also, can you find a way to explain the name of the theorem?

The sought quantity $\mathbb{E}(A_n)$ may be naïvely expressed as:

$$\mathbb{E}(A_n) = \sum_{P \subseteq L} p_P \cdot \text{Area(convex hull of } P)$$

where L is the set of $(n+1)^2$ lattice points on the geoboard and p_P is the probability of the subset of holes P (and none else) containing the pins after the procedure.

[3] In other words, can Buffon exchange his needles for noodles?

[4] Project Euler problem 514

[5] That is, the cardinality of the set $\{\{\text{pin}_1, \text{pin}_2\} \mid \text{pin}_1 \neq \text{pin}_2 \text{ are two pins placed by Simon onto the geoboard}\}$.

Suggestion 25. Is the above formula practical for hands-on calculation? For an implementation on a computer? If you are familiar with the $\mathcal{O}$-notation, you can estimate the complexity using this.

There are, however, alternative approaches...

Suggestion 26. Given a pair of pins on the grid, how can you determine if the edge between them is part of the convex hull (= the shape S)?

Suggestion 27. Having seen the Shoelace theorem, try to make sense of the following expression:

$$\mathbb{E}(A_n) = \frac{1}{2} \sum p \cdot V \times W$$

What are we summing over? What is V, W, p?

Suggestion 28. Derive the latter formula from the formula in Suggestion 25.

Suggestion 29. Compare the practicality of this expression with the formula in Suggestion 25.

Suggestion 30. Estimate $\mathbb{E}(A_n)$ for small values of n via simulation on a computer (you can simualate the whole procedure multiple times and average the obtained areas of S). There are widely available and google-able algorithms on finding the convex hull of a set of points, for example a built-in version within SciPy[6].

Suggestion 31. How would you implement one of the formulas mentioned above on a computer, obtaining thus exact values? If you can, do this.

Theorem (Pick's theorem)**.** Let P be a convex polygon with vertices at integer coordinates, i be the number of its interior lattice points, and b the number of its boundary lattice points. Then the area A of P is:

$$A = i + \frac{b}{2} - 1$$

Suggestion 32. Give a proof of the Pick's theorem.

Suggestion 33. Pick's theorem gives an alternative to a Shoelace formula for calculating an area of a polygon. Try to find an expression for $\mathbb{E}(A_n)$ using the Pick's theorem instead.

The above methods for finding $\mathbb{E}(A_n)$ are not the only ones, and by no means the most efficient ones either. You can try to look for your own observations and solutions.

Suggestion 34. Let $\text{per}_n(S)$ be, analogously to area, the perimeter of the final shape S. Try to come up with methods for finding $\mathbb{E}(\text{per}_n)$.

St. Petersburg Paradox

At the annual meeting of coin flippers, you are offered a participation in the following game of chance:

- The entry fee to play the game is n pounds.
- In the beginning of the game, kn pounds are put into the pot.
- You start flipping a (fair) coin. If the outcome is a head, the amount p in the pot is updated to $f(p)$. Otherwise, you take whatever is in the pot and the game ends.

Suggestion 35. Suppose $n = 1000$, $k = \frac{1}{10^9}$, $f(p) = 2p$. Would you play the game?

Suggestion 36. What is the highest value of n that you would pay to play the game?

Suggestion 37. Suppose you had a finite amount of money before you start playing the game. How many rounds would you (be able to) play?

Suggestion 38. What mathematical concept would you use to guide your willingness to take part in such games? Is this reasonable[7]?

[6] https://docs.scipy.org/doc/scipy-0.19.0/reference/generated/scipy.spatial.ConvexHull.html
[7] There is lots of philosophy about making decisions. Feel free to explore into this direction!

Suggestion 39. What happens if the coin is not fair?

Suggestion 40. What changes in the game if we assume the the available resources (money) are finite?

Suggestion 41. What is a general expression of the expected value in terms of n, k, f?

Suggestion 42. Analyze the game with different values of n, k, f. You can make plots of expected value with respect to the three variables.

Suggestion 43. If it is within your capabilities, you can write simulations for different values of n, k, f and/or any other variations of the game.

Gambling

Suggestion 44 (Gambler's ruin)**.** Let there be two players possessing n_1, n_2 pounds respectively. A fair coin is being flipped indefinitely; at each flip the loser transfers one pound to the winner. When one of the players has no money left, the game ends.
What is the probability that the game doesn't end?
What are the chances of each player to win the game?

Suggestion 45. You take the role as an evil game designer that target non-mathematicians. Design a game which looks attractive to participants, but which will make you rich in the long run.

Participant evaluation

Thank you so much for participating in the Problem Solving Challenge 2018! We hope that you enjoyed it and that it was worth your time.

In order to improve for coming events in the future we, the organisers, would like to ask you a few questions. These will be anonymous.

- What did you enjoy about the event?
- What could be improved about the event?
- Were the organisers helpful?
- How challenged were you today?
- How difficult were the problems you did today? Too easy? Too hard?
- Did you feel like you could participate in the event given your mathematical background?
- Was the event like you expected? If not, why?
- Do you have any other comments?

Please write your answers either on this sheet or on a separate sheet of paper.