Problem Solving Maths Group
Systems of non-linear equations

Daniel Eriksson

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You have most likely solved a couple of linear systems of equations in your life. If solutions exist, their exist efficient methods for finding those. In this session, we will tackle nonlinear system of equations. No general solution methods exist and finding solutions is in general a hopeless business. Here we will look at some tricks to solve specific such systems. As for linear equations, substitution and adding/subtracting/multiplying equations is useful. Recall algebraic identities

$$(a \pm b)^2 = a^2 \pm 2ab + b^2, \ (a+b)(a-b) = a^2 - b^2$$

$$a^n - b^n = (a-b)(a^{n-1} + ab^{n-2} + \cdots + b^{n-1})$$

$$\log(ab) = \log a + \log b, \ \log(a^b) = b \log a.$$

1. Solve $\begin{cases} x^2 + y^2 = 9 \\ x^2 - y^2 = 7 \end{cases}$ for reals (do you see the problem graphically?).

2. Find integers a, b which satisfies
$$\begin{cases} \frac{1}{a} + \frac{1}{b} = -\frac{1}{2} \\ ab = -16 \end{cases}.$$

3. Determine $x^2 + y^2 + z^2$, where x, y, z are integers, given the following system
$$\begin{cases} x + y + z = 60 \\ (x - 4y)^2 + (y - 2z)^2 = 2 \end{cases}$$
[Recall $x^2 \geq 0$ for any x.]

4. Show that
$$\begin{cases} y = \sqrt{x + \sqrt{1-x}} \\ y = \sqrt{y + \sqrt{1+y}} \end{cases}$$
has no real solutions.

5. Find an integer k such that
$$\begin{cases} i^k = i \\ e^{-\frac{k}{9}} = i^{\frac{2i}{\pi}}, i^2 = -1 \end{cases}$$
where $i^2 = -1$.
Hint: how can i be written as an exponential?

6. Find all non-negative solutions to
$$\begin{cases} x^2 - yz = x \\ y^2 - zx = y \\ z^2 - xy = z \end{cases}.$$
[Hint $(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + xz + yz)$.]

7. Find the unique real solution to the system
$$\begin{cases} x^y = y^x \\ \frac{y}{x} = \pi \end{cases}.$$
[Hint: logarithms!]

8. Find all real solutions to
$$\begin{cases} x + y - z = 2 \\ x^2 + y^2 = z^2 \\ xyz = 60 \end{cases}.$$

9. Find all natural solutions to
$$\begin{cases} abc = 2016 \\ (a-1)(b-1)(c-1) = 1573 \end{cases}.$$