Tiling

Miroslav Stankovic

Problem Solving Group 2016/17
Semester 2 - Week 3

Tiling problems are about covering a space (usually a plane or its part), using specific set of tiles. Tiling problems arise in different areas of mathematics, but are usually considered as problems in Combinatorics or Algebra. In this paper, we will be mostly looking at a finite two-dimensional grid to be covered by square tiles, *polyomino* tiles, or (if you feel Escheresque today) by rep tiles.

1 Tiling for painters

The traditional approach to tiling problems is using coloring. It is probably best to introduce this technique with an example:

Example 1. *Two opposite corners of an* 8×8 *chessboard are cut off. Is it possible to cover the remaining squares by* 2×1 *dominos?*

Proof. Each domino covers two adjacent chessboard squares, thus exactly one white and one black square. The dominos therefore cover the same number of black and white squares. But we have removed two squares of same color, leaving us with unequal number of white and black squares to be covered - thus the board cannot be covered using dominos. □

1.1 Coloring problems

Problem 1. *A rectangle is covered by* 2×2 *and* 1×4 *tiles. One tile is replaced by one of the other kind. Show that the rectangle cannot be covered using the new set of tiles.*

Problem 2. *Is it possible to cover a* $10 \times 10 \times 10$ *box using* $1 \times 1 \times 4$ *tiles?*

Problem 3. *Which single square can be removed from a* 7×7 *board so that the rest can be tiled with* 1×3 *trominos?*

2 Tiling for non-combinatorialists

In this section, we will look at a completely different approach to solving tiling problems. This method uses basic physics and elementary number theory.

Definition 1. *The centroid of a system of points* $X_1, \dots, X_n$ *with masses* $m_1, \dots, m_n$ *is a point* O *which satisfies*

$$m_1\overrightarrow{OX_1} + \cdots + m_n\overrightarrow{OX_n} = \overrightarrow{0} \tag{1}$$

Exercise 1. *Show that if the centroid exists, it is unique.*

Theorem 1 (Mass regrouping)**.** *The centroid of a system of points does not change if a subset of the points is replaced by one point situated in their centroid and whose mass is equal to the sum of their masses.*

Proof. As exercise. □

Example 2. *Two opposite corners of an* 8×8 *chessboard are cut off. Is it possible to cover the remaining squares by* 2×1 *dominos?*

Proof. A domino is composed of two adjacent squares, with its centroid in the middle of their shared side.

Let's place the chessboard to plane with origin of our coordinate system placed in the middle (centroid) of the chessboard, while the unit length selected is half of the square size. Using this system of coordinates gives us some very helpful properties.

1. A centroid of the chessboard is at $\vec{0}$.
2. Centroids of individual dominos are integers.
3. Centroids of dominos have one even and one odd coordinate, thus sum of the two coordinates is odd.

Now, since we need 31 dominos to cover the chessboard, the totoal sum of the dominos' coordinates, and thus sum of centroid coordinates is odd. This contradicts that centroid is $\vec{0}$. So no. It is not possible. □

2.1 Centroid problems

Problem 4. *Show that a* 10×10 *chessboard cannot be covered by* 25 *straight tetrominos.*

Problem 5. *A rectangle is tiled with a set of square and straight tetrominos. Prove that if one tile is replaced with one of the other kind, rectangle cannot be tiled with this new set of tiles.*

Problem 6. *Show that a* 10×10 *chessboard cannot be covered by* 25 *T-tetrominos.*

Problem 7. *Consider an* $n \times n$ *chessboard with the four corners removed. For which values of* n *can you cover the board with L-tetrominos?*

Problem 8. *Consider an* $n \times n$ *board tiled with T-tetrominos. Let a,b,c,d, respective be the number of tetrominos of shape* $\top$, $\bot$, $\dashv$, $\vdash$, *respectively. Prove that* $4|(a + b - c - d)$.

Problem 9. *Prove it is possible to fill (completely and without remainder) the box of size* $a \times b \times c$ *with cuboids of size* $4 \times 1 \times 1$ *if and only if at least two of the rectangles* $a \times b$, $a \times c$, $b \times c$ *can be tiled completely and without remainder with tiles of size* 4×1.

3 Tiling for calculators

In most of the previous problems we have been proving non-existence of tiling. Here we look at possible tilings, and out goal is to find how many of them are there.

3.1 Counting problems

Problem 10. *In how many ways can a* $1 \times n$ *rectangle be covered using* 1×1 *and* 1×2 *tiles?*

Problem 11. *In how many ways can a* $2 \times n$ *rectangle be covered using domino tiles?*

Problem 12. *In how many ways can a* $2 \times 3 \times n$ *rectangle be covered using* $2 \times 2 \times 3$ *and* $1 \times 2 \times 3$ *tiles?*

4 Extra Problems

This section contains some other problems, which may or may not be solvable using techniques above.

Problem 13. *Show that if a rectangle can be tiled by smaller rectangles each of which has at least one integer side, then the tiled rectangle has at least one integer side.* [1]

Problem 14. *A* 6×6 *board is tiled by dominoes. Show that there is a line that cuts the board into two parts without cutting any domino.*

Problem 15. *A regular hexagon is divided into a triangular grid, and completely tiled with diamonds (two triangles glued together). Diamonds can be placed in one of three orientations. Prove that, no matter how the board is tiled, there will be the same number of diamonds in each orientation.*

Problem 16. *Consider the region* $T(n)$ *consisting of a triangular array of* $n(n+1)/2$ *unit regular hexagons. For which* n *can* $T(n)$ *be covered by tiles of shape* $T(2)$*?*

References

[1] Harun Šiljak. *How Many Non-combinatorialists Does it Take to Solve a Tiling Problem?*. Jurnalul Matematic Aradean, Vol. 3, No. 1, Jan. 2011.

[2] Stan Wagon.*Fourteen Proofs of a Result About Tiling a Rectangle.* Amer. Math. Monthly, 94 (1987) 601–617.

[1] There is a paper [2] that contains 14 proofs of this problem.